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Cell-pattern sensitivity to box configuration in a saturated porous medium

By B. BORKOWSKA-PAWLAK AND W. KORDYLEWSKI
Instytut Techniki Cieplnej i Mechaniki Plynow, 50-370 Wroclaw, Poland

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Steady small-amplitude thermal convection in a fluid-saturated, infinitely extended porous layer is investigated theoretically in the wavenumber range $1/\surd 2$–1. It was shown that the point of multiple bifurcation $Ra_0 = (3/\surd 2+2)\pi^2$, $\alpha_0 = 2^{-0.25}$ leads to secondary bifurcation when the wavenumber decreases.

As a result a new branch of a stable, complicated, three-dimensional flow in the square cell was discovered for α close to α_0. This branch joins two adjacent branches of three-dimensional flows emanating from the trivial solution and causes their stability transition at the branching points.

1. Introduction

Although natural convection in a porous medium has received considerable attention since the earliest work by Horton & Rogers (1945), there are still some points of interest and contention. The critical Rayleigh number $Ra = 4\pi^2$ for the onset of convection in porous media was determined by Lapwood (1948). Infinitesimal convection occurring in porous media has been studied for Rayleigh numbers close to $4\pi^2$ by means of a perturbation method in two-dimensional physical space by Palm, Weber & Kvernvold (1972) and by Joseph (1976), and in three-dimensional space by Zabib & Kaseoy (1976).

Recently Rudraiah & Srimani (1980) have made an extensive analysis of the physically preferred cell pattern for convection in a porous medium in the spirit of Malkus & Veronis (1958). They observed that at $Ra = 4\pi^2$ three-dimensional flow of square cells with wavenumber $\alpha = 1/\surd 2$ and two two-dimensional rolls with $\alpha = 1$ can occur.

We have completed their studies by an analysis of small-amplitude solutions of the Darcy–Boussinesq equations for wavenumbers in the range 1–$1/\surd 2$. It was found that $Ra_0 = (3/\surd 2+2)\pi^2$, $\alpha_0 = 2^{-0.25}$ is the point of multiple bifurcation and that a variation of the wavenumber α leads to secondary bifurcation. New branches of stable and unstable three-dimensional flows were discovered for Rayleigh numbers close to Ra_0. We have examined the local influence of the difference of the horizontal wavenumbers on the existence and structure of the solutions.

2. Formulation of the problem

We consider a rectangular box of fluid-saturated porous material heated from below. The horizontal plates of the box are non-permeable and perfectly insulating. Heat transfer and fluid motion are described by the Darcy–Boussinesq equations presented here in the non-stationary dimensionless form

$$-\boldsymbol{u}-\nabla p+Ra\,\theta\boldsymbol{k} = 0, \quad \nabla\cdot\boldsymbol{u} = 0, \tag{2.1}$$

$$\frac{\partial \theta}{\partial t} = \nabla^2 \theta + u_z - \boldsymbol{u} \cdot \nabla \theta, \tag{2.2}$$

with the following boundary conditions:

on the upper and lower planes

$$u_z = 0, \quad \theta = 0 \quad (z = 0, 1); \tag{2.3}$$

on the sidewalls

$$u_x = 0, \quad \frac{\partial \theta}{\partial x} = 0 \quad \left(x = 0, \frac{1}{\alpha_1}\right); \tag{2.4a}$$

$$u_y = 0, \quad \frac{\partial \theta}{\partial y} = 0 \quad \left(y = 0, \frac{1}{\alpha_2}\right); \tag{2.4b}$$

where $\boldsymbol{u} = (u_x, u_y, u_z)$, θ and p denote the velocity vector, temperature and pressure respectively, and $\boldsymbol{k}$ is the vertical unit vector. The Rayleigh number Ra is defined in Gupta & Joseph (1972), where it was denoted by R, and α_1, α_2 are the wavenumbers.

Straus & Schubert (1979) showed that it is more convenient to replace (2.1) by a single equation for the potential Φ defined by

$$u_x = \frac{\partial^2 \Phi}{\partial z\, \partial x}, \quad u_y = \frac{\partial^2 \Phi}{\partial z\, \partial y}, \quad u_z = -\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2}.$$

Hence the system of equations (2.1) and (2.2) assumes the form

$$\nabla^2 \Phi = -Ra\, \theta, \tag{2.5}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial^2 \Phi}{\partial x\, \partial z}\frac{\partial \theta}{\partial x} + \frac{\partial^2 \Phi}{\partial y\, \partial z}\frac{\partial \theta}{\partial y} - \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right)\frac{\partial \theta}{\partial z} = \nabla^2 \theta - \frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2}. \tag{2.6}$$

The boundary conditions on θ and Φ are

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad (z = 0, 1),$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial^2 \Phi}{\partial x\, \partial z} = 0 \quad \left(x = 0, \frac{1}{\alpha_1}\right),$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial^2 \Phi}{\partial y\, \partial z} = 0 \quad \left(y = 0, \frac{1}{\alpha_2}\right).$$

Further, (2.5) and (2.6) can be reduced to a single equation for the potential Φ:

$$\frac{\partial \nabla^2 \Phi}{\partial t} + \frac{\partial^2 \Phi}{\partial x\, \partial z}\nabla^2 \frac{\partial \Phi}{\partial x} + \frac{\partial^2 \Phi}{\partial y\, \partial z}\nabla^2 \frac{\partial \Phi}{\partial y} - \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right)\nabla^2 \frac{\partial \Phi}{\partial z} = \nabla^4 \Phi + Ra\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right), \tag{2.7}$$

with the following boundary conditions:

on the horizontal planes

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad (z = 0, 1); \tag{2.7a}$$

on the sidewalls

$$\frac{\partial^2 \Phi}{\partial x\, \partial z} = \nabla^2 \frac{\partial \Phi}{\partial x} = 0 \quad \left(x = 0, \frac{1}{\alpha_1}\right); \tag{2.7b}$$

$$\frac{\partial^2 \Phi}{\partial y\, \partial z} = \nabla^2 \frac{\partial \Phi}{\partial y} = 0 \quad \left(y = 0, \frac{1}{\alpha_2}\right). \tag{2.7c}$$

The linearization of (2.7) for the trivial solution leads to the linear self-adjoint eigenvalue problem

$$\nabla^4\phi + Ra\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 0, \tag{2.8}$$

with homogenous boundary conditions corresponding to (2.7*a–c*). The eigenvalues of (2.8) have the form

$$Ra = \pi^2\frac{[(j\alpha_1)^2+(m\alpha_2)^2+n^2]^2}{(j\alpha_1)^2+(m\alpha_2)^2}, \tag{2.9}$$

and the corresponding eigenfunctions are given by

$$\phi = \sin(n\pi z)\cos(j\pi\alpha_1 x)\cos(m\pi\alpha_2 y). \tag{2.10}$$

The first eigenvalue is the critical Rayleigh number for the onset of convection in a porous layer. Ra assumes the minimal value $4\pi^2$ when $n = m = j = 1$ and for the following wavenumbers:

$\alpha_1 = 1, \quad \alpha_2 = 0$ (two-dimensional rolls),

$\alpha_1 = 0, \quad \alpha_2 = 1$ (Ra is a double eigenvalue),

$\alpha_1 = \alpha_2 = 1/\surd 2$ (three-dimensional square-cell, Ra is a single eigenvalue).

It is an interesting fact that the second eigenvalue is $Ra = 4.5\pi^2$ and has the following eigenfunctions of the form (2.10), also for $m = n = j = 1$ depending on the wavenumbers:

$\alpha_1 = 1/\surd 2, \quad \alpha_2 = 0$ (two-dimensional rolls),

$\alpha_1 = 0, \quad \alpha_2 = 1/\surd 2$ (Ra is a double eigenvalue),

$\alpha_1 = \alpha_2 = 1$ (three-dimensional square cell, Ra is a single eigenvalue).

For $\alpha_0 = 2^{-0.25}$ the first and second eigenvalues coalesce at $Ra_0 = (3/\surd 2+2)\pi^2$, hence (Ra_0, α_0) becomes a triple bifurcation point with null-space N:

$$\left.\begin{aligned} \phi_1 &= \sin(\pi z)\cos(\pi\alpha_0 x), \\ \phi_2 &= \sin(\pi z)\cos(\pi\alpha_0 y), \\ \phi_3 &= \sin(\pi z)\cos(\pi\alpha_0 x)\cos(\pi\alpha_0 y). \end{aligned}\right\} \tag{2.11}$$

An important illustration of our considerations is given by Straus & Schubert (1981) because the point (Ra_0, α_0) relates to the intersection point of the curves (*a*) and (*f*) in their figure 2. This figure gives also examples of other points of multiple bifurcation, but they correspond to higher modes.

The trivial solution loses the stability of the smallest critical Rayleigh number, and a new steady-state solution appears. It is the purpose of the following sections to derive these solutions and to analyse their stability.

3. Analysis

Recently Coullet & Spiegel (1982) specified two general approaches to the bifurcation problem in thermal convection: namely asymptotic procedures and the direct method of modal expansion and reduction to normal form. However, the bifurcation problem under consideration is degenerate, and hence both of these approaches would lead to long and complicated calculations. We propose to use the Galerkin method with controlled series truncation. We expand the solution Φ of the problem (2.7) in the Fourier series

$$\Phi = \sum_{n,j,m}^{\infty} \Phi_{njm} F_{njm}, \tag{3.1}$$

where

$$F_{njm} = \begin{cases} \sqrt{2}\sin(n\pi z) \quad (j = m = 0), \\ 2\sqrt{\alpha}\sin(n\pi z)\cos(\alpha j\pi x) \quad (j \neq 0, m = 0), \\ 2\sqrt{\alpha}\sin(n\pi z)\cos(\alpha m\pi y) \quad (j = 0, m = 0), \\ 2\sqrt{2\alpha}\sin(n\pi z)\cos(\alpha j\pi x)\cos(\alpha m\pi y) \quad (j \neq 0, m \neq 0). \end{cases}$$

The dimension of the finite Galerkin space will be chosen in such a way that the second-order approximation of small amplitude steady-state solutions will be secured. In order to explain the choice of the appropriate modes, consider (2.7) in the general form

$$\mathrm{F}(\Phi, Ra) = 0 \tag{3.2}$$

Assume an approximate solution $\tilde{\Phi}$ of (3.2) in the form (see Appendix)

$$\tilde{\Phi} = \epsilon\Phi_I + \epsilon^2\Phi_{II}, \quad Ra = Ra_0 + \epsilon^2, \tag{3.3}$$

where

$$\Phi_I = \beta_1\phi_1 + \beta_2\phi_2 + \beta_3\phi_3. \tag{3.4}$$

The task is to determine the coefficients $\beta_1, \beta_2, \beta_3$ and the function Φ_{II}.

The function Φ_{II} follows from the equation of the second perturbation

$$\mathrm{F}_{\Phi}\,\Phi_{II} = \tfrac{1}{2}\mathrm{F}_{\Phi\Phi}\,\Phi_I\,\Phi_I. \tag{3.5}$$

The right-hand side of the above equation is in this case the linear combination of the following modes:

$$\Phi_{200}, \quad \Phi_{210}, \quad \Phi_{201}, \quad \Phi_{211}, \quad \Phi_{212}, \quad \Phi_{221}, \quad \Phi_{220}, \quad \Phi_{202}. \tag{3.6}$$

Hence the finite basis for the Galerkin method providing the second-order approximation of the small-amplitude solutions of (2.7) contains the eigenmodes (110, 101, 111) and additionally (200, 210, 201, 211, 212, 221, 220, 202). As a result we get eleven differential equations in the form

$$\left.\begin{aligned} \frac{\mathrm{d}\Phi_{110}}{\mathrm{d}t} &= a\Phi_{110} + \dots, \\ \frac{\mathrm{d}\Phi_{101}}{\mathrm{d}t} &= a\Phi_{101} + \dots, \\ \frac{\mathrm{d}\Phi_{111}}{\mathrm{d}t} &= e\Phi_{111} + \dots, \\ &\vdots \end{aligned}\right\} \tag{3.7}$$

where ... denotes nonlinear terms.

First we consider the steady-state solutions of the system (3.7). Eliminating the modes (3.6) corresponding to the second-order term, we obtain the following system of three cubic equations for the unknowns $x = \Phi_{110}$, $y = \Phi_{101}$, $z = \Phi_{111}$:

$$\left.\begin{aligned} x(-a + bx^2 + cy^2 + dz^2) &= 0, \\ y(-a + cx^2 + by^2 + dz^2) &= 0, \\ z(-e + fx^2 + fy^2 + gz^2) &= 0, \end{aligned}\right\} \tag{3.8}$$

where

$$\left.\begin{aligned}
a &= \frac{\alpha^2 Ra - \pi^2(1+\alpha^2)^2}{1+\alpha^2},\\
e &= \frac{2\alpha^2 Ra - \pi^2(1+2\alpha^2)^2}{1+2\alpha^2},\\
b &= \tfrac{1}{2}\alpha^6\pi^4,\\
c &= \frac{\alpha^6\pi^4}{2}\left[1-\frac{2\pi^2(2+\alpha^2)}{\alpha^2 Ra - 2\pi^2(2+\alpha^2)^2}\right],\\
d &= \alpha^6\pi^4\left[\frac{1+2\alpha^2}{1+\alpha^2}-\frac{\pi^2(13+5\alpha^2)\,(5+7\alpha^2)}{2(1+\alpha^2)\,(\alpha^2 Ra-\pi^2(4+\alpha^2)^2)}-\frac{16\pi^2(5+7\pi^2)}{5\alpha^2 Ra-\pi^2(4+5\alpha^2)^2}\right]\\
f &= \frac{\alpha^6\pi^4(1+\alpha^2)}{1+2\alpha^2}\left[1-\frac{\pi^2(7+\alpha^2)\,(7\alpha^2+5)}{2(1+\alpha^2)\,(\alpha^2 Ra-\pi^2(4+\alpha^2)^2)}+\frac{16\pi^2(1+\alpha^2)}{5\alpha^2 Ra-\pi^2(4+5\alpha^2)^2}\right],\\
g &= 2\alpha^6\pi^4\left[1-\frac{64\pi^2(1+\alpha^2)}{\alpha^2 Ra - 4\pi^2(1+\alpha^2)^2}\right].
\end{aligned}\right\} \quad (3.9)$$

The resulting system can be expressed as a set of predator–prey equations and has been studied by biologists (May & Leonard 1975).

In §4 we shall analyse the properties of the solutions of these equations and their stability.

4. Solutions for rolls and cellular cells

It is easy to establish that (3.8) have the following set of solutions:

(i) $x^2 = \dfrac{a}{b}, \quad y = z = 0,$

(ii) $y^2 = \dfrac{a}{b} \quad x = z = 0,$

(iii) $x^2 = y^2 = \dfrac{a}{b+c}, \quad z = 0,$

(iv) $z^2 = \dfrac{e}{g}, \quad x = y = 0,$

(v) $x^2 = y^2 = \dfrac{ag-ed}{J}, \quad z^2 = \dfrac{(b+c)\,e-2fa}{J},$

where $J = (b+c)\,g - 2df$,

(vi) $y^2 = \dfrac{ag-ed}{bg-df}, \quad x = 0, \quad z^2 = \dfrac{eb-af}{bg-df},$

(vii) $x^2 = \dfrac{ag-ed}{bg-df}, \quad y = 0, \quad z^2 = \dfrac{eb-af}{bg-df}.$

The solutions (i) and (ii) correspond to two-dimensional rolls, the solution (iii) describes a three-dimensional flow that is a superposition of perpendicular rolls. The solutions (iv)–(vii) are also three-dimensional flows.

Further, we consider a near neighbourhood of the point (Ra_0, α_0). Because the coefficients b, c, d, f, g and the Jacobian J are positive and smooth functions of the wavenumber and of the Rayleigh number in the neighbourhood of (Ra_0, α_0), we omit the standard perturbation analysis and calculate these coefficients at (Ra_0, α_0):

$$\left.\begin{aligned} b &= 0.177\ldots\pi^4, & c &= 0.257\ldots\pi^4, \\ d &= 2.718\ldots\pi^4, & f &= 0.38\ldots\pi^4, \\ g &= 9.543\ldots\pi^4, & J &= 2.09\ldots\pi^8. \end{aligned}\right\} \tag{4.1}$$

Therefore we consider the dependence of x, y, z and Ra only on a and e. Not all solutions exist for any value of wavenumber α and of the Rayleigh number Ra. The branches of solutions (i)–(iii) appear when the Rayleigh number crosses the first eigenvalue $\pi^2(1+\alpha^2)^2/\alpha^2$. The fully three-dimensional solution (iv) can exist when the Rayleigh number exceeds the second eigenvalue $\pi^2(1+2\alpha^2)^2/2\alpha^2$. The solutions (v), (vi) and (vii) bifurcate with non-trivial solutions (iii), (ii) and (i) (the secondary bifurcation). Elementary calculations provide the coordinates of the points of secondary bifurcation:

$$Ra_1 = A\frac{g-Bd}{Bg-2d}, \quad x = y = 0, \quad z^2 = \frac{e}{g},$$

$$Ra_2 = A\frac{2f-B(b+c)}{2Bf-2(b+c)}, \quad x^2 = y^2 = \frac{a}{b+c}, \quad z = 0,$$

$$Ra_3 = A\frac{f-Bb}{Bf-2b}, \quad x^2 = \frac{a}{b}, \quad y = z = 0,$$

where

$$A = \frac{\pi^2(1+\alpha^2)(1+2\alpha^2)}{\alpha^2} \quad B = \frac{1+2\alpha^2}{1+\alpha^2}.$$

The existence of branches of solutions (v)–(vii) requires that the wavenumber should satisfy $\alpha < \alpha_0$. These solutions disappear at the secondary bifurcation points, i.e. (vi) and (vii) at Ra_2 and (v) at Ra_3. For Rayleigh numbers larger than Ra_3 solutions (i)–(iv) exist; however, only two-dimensional rolls are stable. To illustrate these considerations we present (figure 1 *a*) the dependence of the solutions (i)–(vii) on the Rayleigh number for the wavenumber $\alpha = \sqrt{0.7}$ calculated numerically with the use of coefficients (4.1).

The values Ra_1, Ra_2 and Ra_3 are

$$Ra_1 = 4.138\ldots\pi^2, \quad Ra_2 = 4.156\ldots\pi^2, \quad Ra_3 = 4.190\ldots\pi^2.$$

When the wavenumber increases to α_0, the Rayleigh numbers Ra_1, Ra_2 and Ra_3 tend to Ra_0, and at $\alpha = \alpha_0$ the branches (v)–(vii) vanish. For wavenumbers α larger than α_0 there are no branches (v)–(vii) and no points of secondary bifurcation Ra_1, Ra_2 and Ra_3. The dependence of the solutions (i)–(iv) on the Rayleigh number for $\alpha = \sqrt{0.71}$ is presented in figure 1 (*b*). A comparison of figures 1 (*a*) and (*b*) suggests that the flow pattern in the box filled by porous material can be essentially influenced by the box dimensions for Rayleigh numbers close to critical.

The results of the stability analysis of the branches (i)–(vii) are presented in §5.

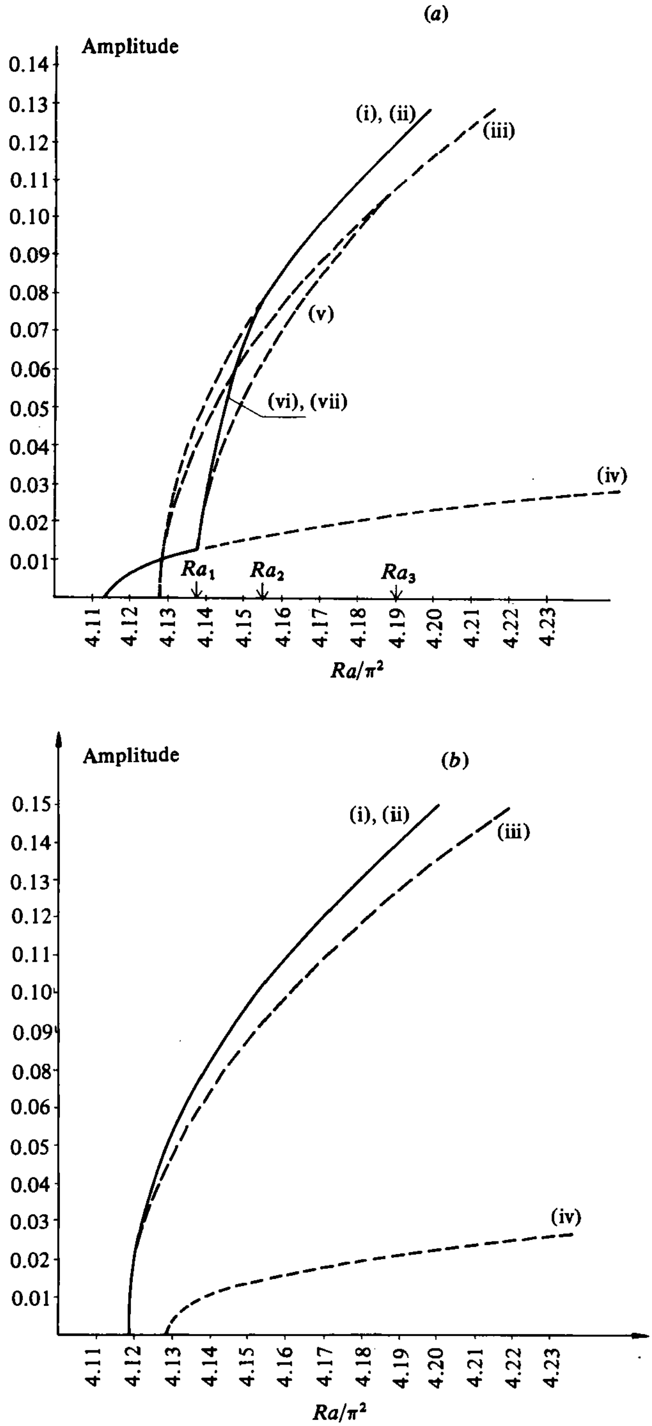

FIGURE 1. (*a*) Amplitudes of the solutions (i)–(vii) *vs.* the Rayleigh number for $\alpha = \sqrt{0.7}$: ——, stable; ---, unstable solutions. (*b*) Amplitudes of the solutions (i)–(iv) *vs.* the Rayleigh number for $\alpha = \sqrt{0.71}$: ——, stable; ---, unstable solutions.

5. Stability

In order to analyse the stability of small-amplitude solutions we will return to (3.7). The expansion (3.3) suggests that the Fourier coefficients Φ_{110}, Φ_{101}, Φ_{111} are $O(\epsilon)$ and that the rest of coefficients are $O(\epsilon^2)$ for small $\epsilon = (Ra - Ra_0)^{\frac{1}{2}}$. Because a and e are $O(\epsilon)$, then, introducing a new timescale $\tau = \epsilon^2 t$, we separate the system (3.7) into two parts: three differential equations for $x = \Phi_{110}$, $y = \Phi_{101}$, $z = \Phi_{111}$ and eight algebraic equations. Eliminating the coefficients (3.7) from the differential equations we get

$$\left.\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\tau} &= -x(-a+bx^2+cy^2+dz^2), \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= -y(-a+cx^2+by^2+dz^2), \\ \frac{\mathrm{d}z}{\mathrm{d}\tau} &= -z(-e+fx^2+fy^2+gz^2). \end{aligned}\right\} \tag{5.1}$$

This system of equations has interesting connections with those found in the literature.

If we restrict our considerations to the two first equations and assume $z = 0$, then the resulted system of equations becomes the same as that obtained by Segel (1962) for the nonlinear interaction of two disturbances in Bénard convection. We also obtained these equations in analysing stability of the two-dimensional Darcy–Boussinesq convection (Borkowska-Pawlak & Kordylewski 1982). Using the phase-plane technique it is not difficult to determine the stability of steady-state flow pattern and the global behaviour of transient motions in this case.

If we introduce new variables

$$\delta = x^2, \quad \rho = y^2, \quad \gamma = z^2, \quad s = 2\tau$$

the original system of equations (5.1) assumes the form

$$\left.\begin{aligned} \frac{\mathrm{d}\delta}{\mathrm{d}s} &= -\delta(-a+b\delta+c\rho+d\gamma), \\ \frac{\mathrm{d}\rho}{\mathrm{d}s} &= -\rho(-a+c\delta+b\rho+d\gamma), \\ \frac{\mathrm{d}\gamma}{\mathrm{d}s} &= -\gamma(-e+f\delta+f\rho+g\gamma), \end{aligned}\right\} \tag{5.2}$$

where δ, ρ and γ are determined only in $\mathbb{R}^3_+$.

These equations have a similar structure to the system of equations analysed by May & Leonard (1975) for the population problem. However, in this case the coefficients do not obey their rules; hence the behaviour of the solutions is different. The system (5.2) has a trap, i.e. in $\mathbb{R}^3_+$ there is a bounded region B such that every solution of (5.2) eventually becomes trapped by B. This conclusion results from the fact that for every fixed value of the parameters a and e the derivatives $\mathrm{d}\delta/\mathrm{d}s$, $\mathrm{d}\rho/\mathrm{d}s$ and $\mathrm{d}\gamma/\mathrm{d}s$ become negative for sufficiently large $\delta+\rho+\gamma$.

In May & Leonard's paper an important role was played by the product $P = \delta\rho\gamma$, which remained asymptotically invariant. In this case P diminishes quickly for a sufficiently large $\delta+\rho+\gamma$ because we have

$$\frac{\mathrm{d}}{\mathrm{d}s}[\ln P] = -[-2a-e+a+b+f(\delta+\rho)+\gamma(2d+g)].$$

We may interpret this as attracting to the fixed points.

A superficial analysis of (5.2) shows that for $a < 0$ and $e < 0$ only the trivial solution is attracting. If $a < 0$ and $e > 0$ ($\alpha > \alpha_0$) then eventually a non-stationary solution is attracted to one of the two-dimensional steady states (i) or (ii). In the opposite case $a > 0$ and $e < 0$ ($\alpha < \alpha_0$) only the solution (iv) is attracting.

The rigorous stability analysis of the steady-state solutions (i)–(vii) was made by calculations of the eigenvalues of the linearized form of (5.2). The results of analysis are shown in figures 1 (a, b).

A predominant role is played by the orthogonal two-dimensional rolls (i) and (ii). The same conclusion follows from the numerical analysis of Straus & Schubert (1979). However, it is not easy to make a more detailed comparison with their works, because they analysed stability of steady finite-amplitude thermal convection for large values of the Rayleigh number.

6. The Nusselt number

We recall the definition of the Nusselt number as the ratio of the actual heat transport and the heat transported only by conduction for the given temperature difference between the hot and cold planes:

$$Nu = 1 - \int_0^{1/\alpha}\int_0^{1/\alpha} \left.\frac{\partial\theta}{\partial z}\right|_{z=0} \mathrm{d}x\,\mathrm{d}y. \tag{6.1}$$

In terms of modes of the Fourier series the Nusselt number is expressed by

$$Nu = 1 - \pi \sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\sum_{m=i}^{\infty} n c_{jm}\theta_{njm} \int_0^{1/\alpha}\int_0^{1/\alpha} \cos(\alpha m\pi y)\cos(\alpha j\pi x)\,\mathrm{d}x\,\mathrm{d}y,$$

where

$$c_{jm} = \begin{cases} \sqrt{2} & (j = m = 0), \\ 2\sqrt{\alpha} & (j \neq 0, m = 0 \quad \text{or} \quad j = 0, m \neq 0), \\ 2\sqrt{2}\,\alpha & (j \neq 0, m \neq 0). \end{cases}$$

This integral is non-vanishing only for $j = m = 0$. Using (2.5) for the temperature θ and the potential Φ, the Nusselt number can be written as

$$Nu = 1 - \frac{\pi^3}{Ra\,\alpha^2}\sum_{n=1}^{\infty} n^3 \Phi_{n00}. \tag{6.2}$$

Because of the truncation of the Fourier series to (3.6), the above expression reduces to the form

$$Nu = 1 - \frac{8\sqrt{2}\,\pi^3}{Ra\,\alpha^2}\Phi_{200}. \tag{6.3}$$

In the case $\alpha = 1$, which was considered by several authors (e.g. Straus & Schubert 1979; Rudraiah & Srimani 1980; Zebib & Kassoy 1978) we have

$$Nu = 1 - \frac{8\sqrt{2}}{Ra}\pi^3\Phi_{200}.$$

For the stable two-dimensional rolls emanating at $Ra = 4\pi^2$, we found that the Nusselt number is given by

$$Nu_{2d} = 1 + 2\frac{Ra - 4\pi^2}{Ra}. \tag{6.4}$$

For the unstable three-dimensional motion (iii) emanating from $Ra = 4\pi^2$, which is superposition of the two-dimensional rolls, the Nusselt number is

$$Nu_{3d} = 1 + \frac{28}{17}\frac{Ra - 4\pi^2}{Ra}. \tag{6.5}$$

The formulae (6.4) and (6.5) indicate that two-dimensional rolls transport more heat than the three-dimensional cell flow.

When $\alpha \neq 1$ the Nusselt numbers for the abovementioned flows are given by the following formulae:

$$Nu_{2d} = 1 + 2\frac{Ra - \pi^2(1+\alpha^2)^2/\alpha^2}{\alpha^3 Ra},$$

$$Nu_{3d} = 1 + 2\frac{\alpha^4 + 6\alpha^2 + 7}{\alpha^4 + 7\alpha^2 + 9}\frac{Ra - \pi^2(1+\alpha^2)^2/\alpha^2}{\alpha^3 Ra}.$$

By comparing the Nusselt numbers for the two- and three-dimensional motions, we conclude that two-dimensional roll configurations are the physically preferred cell pattern, as they transport more heat.

7. The various horizontal wavenumbers

When the wavenumber in the horizontal directions are different then analysis providing the predator–prey equation (3.8) remains much the same. There are some differences, viz the coefficients of x, y and z are

$$a_1 = \frac{\alpha_1^2 Ra - \pi^2(1+\alpha_1^2)^2}{1+\alpha_1^2},$$

$$a_2 = \frac{\alpha_2^2 Ra - \pi^2(1+\alpha_2^2)^2}{1+\alpha_2^2},$$

$$e = \frac{(\alpha_1^2+\alpha_2^2)\, Ra - \pi^2(1+\alpha_1^2+\alpha_2^2)^2}{1+\alpha_1^2+\alpha_2^2}.$$

Consequently in the solutions (i) and (ii) of the new system of equations the expressions in numerators are replaced by a_1 and a_2 respectively – and similarly for the solutions (vi) and (vii). The form of solution (iv) is identical, while (iii) and (iv) are some linear combinations of e, $a_1 + a_2$ and $a_1 - a_2$.

Thus the bifurcation point Ra_1 'divides' into the following points:

Ra_1, bifurcation point of solution (vi);

Ra_1, bifurcation point of solution (vii);

Ra_1, bifurcation point of solution (v).

Similarly, the bifurcation point Ra_2 'divides' into the points

Ra_2, disappearance point of solution (vi);

Ra_2, disappearance point of solution (vii).

This situation is presented in figure 2.

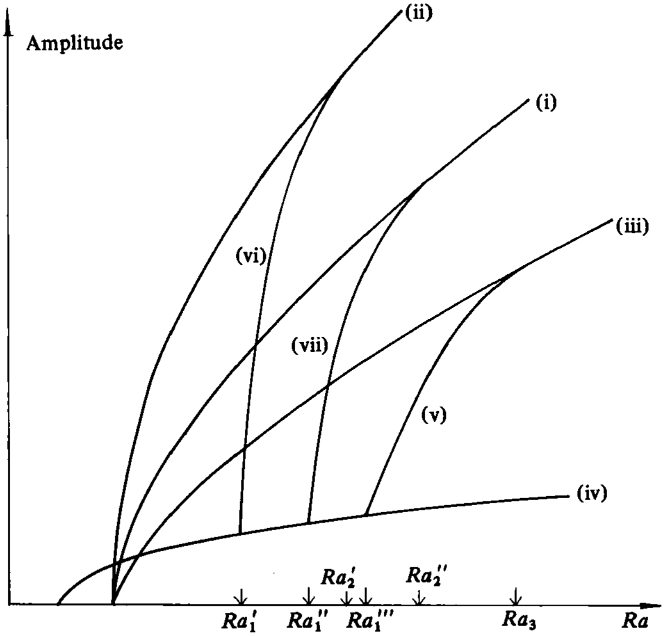

FIGURE 2. Schematic pattern of the amplitudes of the solutions (i)–(vii) *vs.* the Rayleigh number.

8. Summary

It was found that the point of multiple bifurcation $Ra_0 = (3/\surd 2+2)\pi^2, \alpha_0 = 2^{-0.25}$ for the Darcy–Boussinesq equation describing thermal convection in a porous box leads to secondary bifurcations with a small wavenumber rise. As a result, a branch of stationary solutions was discovered that proves that a continuous transition of pattern flows from two-dimensional to three-dimensional structures is possible (and *vice versa*) with Rayleigh-number variation. A similar behaviour of spatial structures due to bifurcation of steady states was also observed in chemical kinetics by Keener (1976).

An analysis of the bifurcation diagrams leads to the conclusion that the box size strongly influences the small-amplitude convective flow in porous media. When the wavenumber increases to α_0, the points of secondary bifurcation Ra_1, Ra_2, Ra_3 tend to Ra_0. When the wavenumber decreases, the point of secondary bifurcation Ra_3 escapes to infinity. For example, numerical calculations not presented in this paper showed that for $\alpha = 1/\surd 2$ there exist only two points of secondary bifurcation Ra_1, Ra_2.

A difference in the horizontal wavenumbers does not induce new solutions; what was observed was only the modification of the structure of the existing solutions. It should be emphasized that the results obtained for fixed α, varying Ra correspond to flow changes in a specific box, while alternating α corresponds to changing the box dimensions.

Appendix

Consider (2.7) in the form

$$\mathrm{F}(\Phi, Ra) = 0 \tag{A 1}$$

where F is an analytical mapping in an appropriate space.

Let $(0, Ra_0)$ be a non-isolated solution of (A 1), suppose that the linear operator $\mathrm{F}_\Phi(0, Ra_0)$ has null space N spanned by $\{\phi_1, \phi_2, \phi_3\}$, and that the null space N^* of $\mathrm{F}^*_\Phi(0, Ra_0)$ is spanned by $\{\phi_1^*, \phi_2^*, \phi_3^*\}$. The operator F has the following properties:

(*a*) $\mathrm{F}(0, Ra) \equiv 0$;

(*b*) $\langle \mathrm{F}_{\Phi\Phi}(0, Ra_0)\,\phi\phi, \phi_i^* \rangle = 0 \quad (i = 1, 2, 3)$,

where $\phi \in N$ and $\langle \psi_1, \psi_2 \rangle = \int_0^{1/\alpha} \int_0^{1/\alpha} \int_0^1 \psi_1 \psi_2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

The bifurcating solutions can be expanded in power series

$$\left.\begin{aligned} Ra &= Ra_0 + \epsilon r_1 + \epsilon^2 r_2 + \dots, \\ \Phi &= \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \dots . \end{aligned}\right\} \tag{A 2}$$

Inserting (A 2) into (A 1) and assuming the Taylor series for $\mathrm{F}(\Phi, Ra)$ yields

$$\mathrm{F}_\Phi(0, Ra_0)\,\Phi_1 = -\mathrm{F}_{Ra}(0, Ra_0)\,r_1, \tag{A 3}$$

$$\mathrm{F}_\Phi(0, Ra_0)\,\Phi_2 = -\tfrac{1}{2}\mathrm{F}_{\Phi\Phi}(0, Ra_0)\,\Phi_1\Phi_1 - \mathrm{F}_{\Phi Ra}(0, Ra_0)\,\Phi_1 r_1 - \mathrm{F}_{Ra}(0, Ra_0)\,r_2. \tag{A 4}$$

Because of (*a*) it follows from (A 3) that $\Phi_1 \in N$; hence we get

$$\Phi_1 = \beta_1\phi_1 + \beta_2\phi_2 + \beta_3\phi_3,$$

where β_1, β_2 and β_3 are constants.

A necessary condition for (A 4) to have a solution is that

$$\tfrac{1}{2}\langle \mathrm{F}_{\Phi\Phi}(0, Ra_0)\,\Phi_1\Phi_1, \phi_i^* \rangle + \langle \mathrm{F}_{\Phi Ra}(0, Ra_0)\,\Phi_1, \phi_i^* \rangle\, r_1 = 0 \quad (i = 1, 2, 3).$$

Because of (*b*) we get

$$r_1 = 0, \tag{A 5}$$

and (A 4) assumes the form

$$\mathrm{F}_\Phi(0, Ra_0)\,\Phi_2 = -\tfrac{1}{2}\mathrm{F}_{\Phi\Phi}\,\Phi_1\Phi_1,$$

where Φ_2 exists and does not belong to N.

Further, the coefficients β_1, β_2, β_3 and r_2 would be calculated from the third-order perturbation equation including the normalization condition

$$\beta_1^2 + \beta_2^2 + \beta_3^2 = 1.$$

In our approach the above condition is neglected; hence $r_2 = 1$, and the approximate solutions include only the first two terms in the expansion (A 2).

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